Quaternionic Electroweak Theory and Cabibbo–Kobayashi–Maskawa Matrix

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We find in our quaternionic version of the electroweak theory an apparently hopeless problem: In going from complex numbers to quaternions, the calculation of the real-valued parameters of the Cabibbo–Kobayashi–Maskawa matrix drastically changes. We aim to explain this *quaternionic puzzle*.

1. INTRODUCTION

In this paper we review some of the basic properties of the quaternionic electroweak theory (De Leo and Rotelli, 1996), based on the one-dimensional local gauge group $U(1, q)_L | U(1, c)_Y$ [quaternionic counterpart of the Glashow (1961) group]. Notwithstanding the recent success in manipulating the non-commutative quaternionic field in quantum mechanics and field theory (Adler, 1986a,b, 1988, 1989, 1994a,b, 1995, 1996b; Adler and Millard, 1995, 1996; De Leo and Rotelli, 1994, 1996b; De Leo, 1996a,b,d), quaternions must be treated with prudence. For example, we meet with a puzzle in our version of the Salam–Weinberg model (Weinberg, 1964; Salam, 1968), namely, *how to reproduce the right calculation of the real-valued parameters of the Cabibbo–Kobayashi–Maskawa (CKM) matrix* (Cabibbo, 1963; Kobayashi and Maskawa, 1973) *by quaternions.*

Historically, the quaternionic field was introduced by Hamilton (1969) in 1843, and after the fundamental contributions to quaternionic quantum mechanics by Finkelstein *et al.* (1962, 1963a,b, 1979) (foundations of quaternionic quantum theories, quaternionic representations of compact groups, etc.) quaternions were somewhat an enigma for physicists. Quaternions was

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restored to life by the work of Horwitz and Biedenharn (1984) (quaternionic tensor product, second quantization, and gauge fields). In the Preface of Adler's (1995) book we read, "In particular, my decision to embark on a detailed investigation of quaternionic quantum mechanics arose both from a question posed to me by Frank Yang and from my study of a preliminary version of the 1984 paper by Larry Biedenharn and Larry Horwitz sent to me by the authors." Today Adler's book represents the main reference one working this research field.

Let us briefly discuss the features of quaternionic numbers. The quaternionic algebra has been expounded in a series of papers (Rembieliński, 1980a,b; Dimitric and Goldsmith, 1989; Razon and Horwitz, 1991a,b, 1992; Nash and Joshi, 1987a,b, 1988, 1993; Horwitz, 1993) and books (Gilmore, 1974; Altmann, 1986); the reader may refer to these for further details. For convenience we repeat and develop the relevant points.

The quaternionic algebra over the real field $\mathcal{R}$ is a set

$$\mathcal{H} = \{\alpha + i\beta + j\gamma + k\delta \,|\, \alpha, \beta, \gamma, \delta \in \mathcal{R}\} \tag{1.1}$$

with the operation of multiplication defined according to the following rules for imaginary units:

$$i^2 = j^2 = k^2 = -1$$

$$ij = k, \qquad jk = i, \qquad ki = j$$

$$ji = -k, \qquad kj = -i, \qquad ik = -j$$

In going from the complex numbers to the quaternions we lose the property of commutativity ($ij \neq ji$). This represents a challenge in manipulating such a numerical field.

Working with noncommutative numbers, we must admit the existence of left and right multiplication; in fact the left action of the operator $\mathcal{O}$ on quaternions q

$$\mathcal{O}q$$

is in general different from its right action

$$q\mathcal{O}$$

In order to distinguish left/right actions, we will use the following terminology for right-acting operators

$$(1 \,|\, \mathcal{O})q \equiv q\mathcal{O}$$

Namely, we introduce the concept of *barred* operators.

Among the favorable results in using *barred* operators, we recall the possibility to reformulate special relativity by quaternions (De Leo, 1996c). Explicitly, the quaternionic generators of the Lorentz group are

$$\text{boost } (ct, x) \quad \frac{k|j - j|k}{2}$$

$$\text{boost } (ct, y) \quad \frac{i|k - k|i}{2}$$

$$\text{boost } (ct, z) \quad \frac{j|i - i|j}{2}$$

$$\text{rotation } around \; x \quad \frac{i - 1|i}{2}$$

$$\text{rotation } around \; y \quad \frac{j - 1|j}{2}$$

$$\text{rotation } around \; z \quad \frac{k - 1|k}{2}$$

The four real quantities which identify the space-time point (ct, x, y, z) are represented by the quaternion

$$q = ct + ix + jy + kz$$

This gives the natural generalization of the Hamilton's (1969) idea. The Irish physicist used quaternions to describe the rotations in the three-dimensional space

$$e^{(iu_x + ju_y + ku_z)\theta/2}(ix + jy + kz)e^{-(iu_x + ju_y + ku_z)\theta/2} \tag{1.2}$$

where $\mathbf{u} \equiv (u_x, u_y, u_z)$ identifies the rotation axis and θ the rotation angle.

This paper is structured as follows: In Section 2 we review some of the basic properties of the quaternionic electroweak theory; in particular, we discuss the complex scalar product, Dirac equation, complex projection of the Lagrangian, quaternionic Higgs field, and one-dimensional local gauge group. In Section 3 we explain our *quaternionic puzzle*, by showing that in our quaternionic version of the Salam–Weinberg model, the CKM matrix must be "complex-barred." In the last section we draw our conclusions.

2. QUATERNIONIC ELECTROWEAK THEORY

An essential ingredient in our version of quaternionic quantum mechanics is what Rembieliński (1978) called the adoption of a complex geometry

(complex scalar products). This choice is certainly less ambitious than that of Adler (1995), who advocates the use of a quaternionic geometry to reformulate a *new* quantum mechanics. Nevertheless we recall that up to a decade ago the use of quaternionic wave functions made the definition of tensor products ambiguous. Complex geometry allows us to overcome many problems due to the noncommutativity of quaternionic numbers.

Let us address the questions of whether and how quaternionic quantum mechanics (QQM) with complex geometry relates to the observed physical world.

2.1. Momentum Operator in QQM

Although there is in QQM an anti-self-adjoint operator ∂ with all the properties of a translation operator, imposing a quaternionic geometry, there is no corresponding quaternionic self-adjoint operator with all the properties expected for a momentum operator. This hopeless situation is also highlighted in Adler (1995, p. 63).

The usual choice $\mathbf{p} \equiv -i\partial$ still gives a self-adjoint operator with the standard commutation relations with the coordinates, but such an operator does not commute with the Hamiltonian, which will be in general a quaternionic quantity. Nevertheless we can overcome such a difficulty using a complex scalar product

$$\langle \psi \,|\, \varphi \rangle_c = \frac{1 - i\,|\,i}{2} \, \langle \psi \,|\, \varphi \rangle \tag{2.1}$$

and defining as the appropriate momentum operator

$$\mathbf{p} \equiv -\partial\,|\,i \tag{2.2}$$

Now the momentum operator (2.2) is *formally real* and so it commutes with a generic quaternionic Hamiltonian; further, by using a complex geometry, it represents self-adjoint operators

$$\langle \psi \,|\, \partial \varphi i \rangle_c = \langle \partial \psi i \,|\, \varphi \rangle_c$$

Complex projections of scalar products were used by Horwitz and Biedenharn (1984) in order to obtain consistently multiparticle quaternionic states. In a recent paper (De Leo and Rotelli, 1995a), we also find an explicit definition of quaternionic tensor product.

2.2. Dirac Equation and Complex-Valued Lagrangian

An interesting application of quaternions in quantum physics is represented by the quaternionic formulation of the Dirac equation (Rotelli, 1989a).

The need to use complex scalar products no longer relies solely on arguments relative to tensor product spaces (multiparticle systems), but is explicit in the single free-particle wave equation.

In order to write relativistically covariant equations we must treat the space components and time in the same way, hence we are obliged to modify the standard equations by the substitution

$$i\partial_t \to \partial_t | i$$

and so the first modification that must be made in rewriting the Dirac equation is

$$\partial_t \psi i = (\boldsymbol{\alpha} \cdot \mathbf{p} + \beta m)\psi \qquad [\mathbf{p} \equiv -\partial | i] \tag{2.3}$$

The Dirac algebra upon the *reals* (but not upon complex) has a two-dimensional irreducible representation with quaternions. Thus the standard 4×4 complex matrices $(\boldsymbol{\alpha}, \beta)$ reduce to 2×2 quaternionic matrices. A particular representation is given by

$$\beta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad \boldsymbol{\alpha} = \mathbf{Q}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \qquad [\mathbf{Q} \equiv (i, j, k)]$$

Notwithstanding the two-component structure of the wave function, *all four standard solutions appear*

$$ue^{-ipx}, \qquad uje^{-ipx}, \qquad ve^{-ipx}, \qquad vje^{-ipx}$$

The trace theorems are modified (Rotelli, 1989b), but the standard electrodynamics is reproduced.

Let us now discuss the use of the variational principle within QQM. This is nontrivial because of the noncommutative nature of quaternions. As a first hypothesis we consider the traditional form for the Dirac–Lagrangian density:

$$\mathscr{L} = \bar{\psi}\gamma^\mu \partial_\mu \psi i - m\bar{\psi}\psi \tag{2.4}$$

The position of the imaginary unit is due to the fact that, in QQM, the ∂_μ operator is more precisely part of the first-quantized momentum operator $\partial_\mu | i$. The previous Lagrangian is not Hermitian; in fact,

$$(\bar{\psi}\gamma^\mu \partial_\mu \psi i)^\dagger \neq \bar{\psi}\gamma^\mu \partial_\mu \psi i$$

The correct form of the kinetic term reads

$$\mathscr{L}_k = \tfrac{1}{2}[\bar{\psi}\gamma^\mu \partial_\mu \psi i - i(\partial_\mu \bar{\psi})\gamma^\mu \psi] \tag{2.5}$$

This modification of equation (2.4) renders our Lagrangian Hermitian. The requirement of hermiticity, however, says nothing about the Dirac mass term

in (2.4). It is here that appeal to the variational principle must be made. A variation $\delta\psi$ in ψ cannot be brought to the extreme right in equation (2.5) because of the imaginary unit in the first half of the expression. The only consistent procedure is to generalize the variational rule that says that ψ and $\bar{\psi}$ must be varied *independently*. We thus apply independent variations to ψ and ψi, respectively $\delta\psi$ and $\delta(\psi i)$. Similarly for $\delta\bar{\psi}$ and $\delta(i\bar{\psi})$. Now to obtain the desired Dirac equation for ψ and its adjoint equation for $\bar{\psi}$, we are obliged to modify the mass term into

$$\mathcal{L}_m = -\frac{m}{2} [\bar{\psi}\psi - i\bar{\psi}\psi i] \tag{2.6}$$

The final result for $\mathcal{L}$ is (De Leo and Rotelli, 1995c)

$$\mathcal{L}_D = \frac{1}{2} [\bar{\psi}\gamma^\mu \partial_\mu \psi i - i(\partial_\mu \bar{\psi})\gamma^\mu \psi] - \frac{m}{2} [\bar{\psi}\psi - i\bar{\psi}\psi i] \tag{2.7}$$

Considering this last equation, we observe that it is nothing other than the complex projection of equation (2.4),

$$\mathcal{L}_D = \frac{1 - i|i}{2} \mathcal{L} \tag{2.8}$$

Indeed, while $\mathcal{L}$ in equation (2.4) is quaternionic and with the modification of $\mathcal{L}_k$ in equation (2.5) Hermitian, the form given in equation (2.7) is purely complex and Hermitian. Obviously we can write down a quaternionic Hermitian Lagrangian and obtain the correct field equations through the *standard* variational principle by limiting $\delta\psi$ to complex variations (notwithstanding the quaternionic nature of the fields). We consider this latter option unjustified and thus select for the formal structure of $\mathcal{L}$ that of equation (2.7).

2.3. Doubling of Solutions in Bosonic Equation

The Dirac equation represents a *desirable* example of the so-called *doubling of solutions* in QQM with complex geometry. Obviously such a doubling of solutions occurs also in the bosonic equations. For example, we find four complex orthogonal solutions for the Klein–Gordon equation, with the result that, in addition to the two normal solutions e^{-ipx} (positive and negative energy), we discover two *anomalous* solutions je^{-ipx}. The physical significance of the anomalous solutions has been a "puzzle" for the authors. Only recently, by a quaternionic study of the electroweak Higgs sector (De Leo and Rotelli, 1995b) have we been able to identify anomalous Higgs particles.

As remarked in the previous subsection, working within QQM, we need to generalize the variational principle. If φ represents a quaternionic field, its variations $\delta\varphi$ and $\delta(\varphi i) \equiv i\delta\bar{\varphi}$ must be treated independently. In fact, we

can vary $\varphi = \varphi_1 + j\varphi_2$ ($\varphi_{1,2}$ complex), leaving $\bar{\varphi} = \varphi_1 - j\varphi_2$ unchanged. The complex projection also applies to the Klein–Gordon Lagrangian; explicitly

$$\mathscr{L}_{KG} = (\partial_\mu \varphi^\dagger \partial^\mu \varphi - m^2 \varphi^\dagger \varphi)_c$$

where

$$(\varphi^\dagger \varphi)_c = \varphi_1^\dagger \varphi_1 + \varphi_2^\dagger \varphi_2$$

represents the quaternionic generalization of the standard term $\varphi_1^\dagger \varphi_1$. The complex projection kills the *unpleasant* pure quaternionic cross term

$$\varphi_1^\dagger j\varphi_2 - \varphi_2^\dagger j\varphi_1$$

To conclude this brief discussion on the quaternionic variational principle [complete details are given in De Leo and Rotelli (1995b, 1996c)] we recall the main results found: *Going from complex to quaternionic fields, we must admit quaternionic variations for our field, but only complex variations for our Lagrangian.*

Since the only fundamental scalar could be the Higgs boson, in order to interpret the anomalous scalars we believe it to be natural to concentrate our attention on the Higgs sector of the electroweak theory. Moreover, the number of Higgs particles, before spontaneous symmetry breaking, is four and this agrees with the number of quaternionic solutions to the Klein–Gordon equation. The Higgs Lagrangian, in the quaternionic electroweak theory, is

$$\mathscr{L}_H = (\partial_\mu \varphi^\dagger \partial^\mu \varphi)_c - \mu^2(\varphi^\dagger \varphi)_c - |\lambda|(\varphi^\dagger \varphi)_c^2 \qquad (2.9)$$

where

$$\varphi \equiv h^0 + jh^+ \qquad [h^0, h^+ \text{ complex fields}]$$

The Lagrangian (2.9) is obviously invariant under the global group

$$U(1, q) \mid U(1, c)$$

which is the quaternionic counterpart of the complex Glashow group

$$SU(2, c) \times U(1, c)$$

2.4. The Quaternionic Local Gauge Group

We wish now to construct a fermionic Lagrangian invariant under the quaternionic group $U(1, q)$. If we consider a single-particle (two-component) field ψ, we have no hope to achieve this. In fact the most general transformation

$$\psi \to f\psi g \qquad (f, g \text{ quaternionic numbers})$$

is right-limited from the complex projection of our Lagrangian and left-limited from the presence of quaternionic (two-dimensional) γ^μ matrices. So

we could only write a Lagrangian invariant under a right-acting complex $U(1, c)$ group. The situation drastically changes if we use a "left-real" (four-component) Dirac equation

$$(\tilde{\gamma}^\mu \partial_\mu \,|\, i - m)\psi = 0$$

where

$$\tilde{\gamma}^\mu \equiv \gamma^\mu\text{-matrices with } i\text{-factors substituted by } 1\,|\,i$$

In this case we could commute the quaternionic phase and restore the invariance under the left-acting quaternionic unitary group $U(1, q)$.

The massless fermionic Lagrangian (for the first generation) in our quaternionic electroweak model reads

$$\mathcal{L}_c^f = (\overline{\psi}_l \tilde{\gamma}^\mu \partial_\mu \psi_l i + \overline{\psi}_q \tilde{\gamma}^\mu \partial_\mu \psi_q i)_c \tag{2.10}$$

with

$$\psi_l = e + jv, \qquad \psi_q = d + ju \qquad (e, v, d, u \text{ complex fermionic fields})$$

This Lagrangian is globally invariant under the following transformations:

- Left-handed fermions:

$$e_L + jv_L \to e^{-(g/2)\mathbf{Q}\cdot\boldsymbol{\alpha}}(e_L + jv_L)e^{(g/2)iY_l^{(L)}\beta}$$

$$d_L + ju_L \to e^{-(g/2)\mathbf{Q}\cdot\boldsymbol{\alpha}}(d_L + ju_L)e^{(g/2)iY_q^{(L)}\beta}$$

- Right-handed fermions:

$$e_R \to e_R e^{(g/2)iY_e^{(R)}\beta}$$

$$d_R + ju_R \to d_R e^{(g/2)iY_d^{(R)}\beta} + ju_R e^{(g/2)iY_u^{(R)}\beta}$$

The *weak-hypercharge* assignments are

$$Y_l^{(L)} = -1, \qquad Y_q^{(L)} = +\tfrac{1}{3}, \qquad Y_e^{(R)} = -2, \qquad Y_d^{(R)} = -\tfrac{2}{3}, \qquad Y_u^{(R)} = +\tfrac{4}{3}$$

In order to construct a local-invariant theory, we must introduce the following quaternionic gauge field:

$$W_\mu = W_\mu^0 + jW_\mu^+ \qquad [W_\mu^0 = (B_\mu + iW_\mu^1)/\sqrt{2}, \quad W_\mu^+ = (W_\mu^2 - iW_\mu^3)/\sqrt{2}] \tag{2.11}$$

$$\mathbf{W}^\mu \text{ for } U(1, q)_L, \qquad B^\mu \text{ for } U(1, c)_Y$$

by the covariant derivatives

$$\mathscr{D}^\mu(e_L + jv_L) \equiv \left[\partial^\mu - \frac{g}{2}\left(i\,|\,W_1^\mu + j\,|\,W_2^\mu + k\,|\,W_3^\mu\right) - \frac{\tilde{g}}{2}\,|\,B^\mu i\right](e_L + jv_L)$$

$$\mathscr{D}^\mu(d_L + ju_L) \equiv \left[\partial^\mu - \frac{g}{2}\left(i\,|\,W_1^\mu + j\,|\,W_2^\mu + k\,|\,W_3^\mu\right) + \frac{\tilde{g}}{6}\,|\,B^\mu i\right](d_L + ju_L)$$

$$\mathscr{D}^\mu u_R \equiv \left(\partial^\mu + \frac{2\tilde{g}}{3}\,|\,B^\mu i\right)u_R$$

$$\mathscr{D}^\mu d_R \equiv \left(\partial^\mu - \frac{\tilde{g}}{3}\,|\,B^\mu i\right)d_R$$

$$\mathscr{D}^\mu e_R \equiv (\partial^\mu - \tilde{g}\,|\,B^\mu i)e_R$$

The substitution $\partial^\mu \to \mathscr{D}^\mu$ in (2.10) makes our Lagrangian locally invariant.

Full details concerning gauge kinetic terms, Yukawa couplings, interactions among gauge bosons and fermions, and symmetry breaking are reported elsewhere (De Leo and Rotelli, 1996a).

3. THE QUATERNIONIC PUZZLE

In our discussion of the electroweak model, we limited the number of fermion generations to one. We now lift that restriction and consider the implication of having N generations. Although the existing experimental situation supports the value $N = 3$, we shall take N arbitrary in our analysis.

3.1. Complex Mixing Matrix

In this subsection we briefly recall the fermion mixing of the standard (complex) Salam–Weinberg model. By convention, the mixing is assigned to the $Q = 1/3$ quarks by

$$J_{ch}^\mu = \bar{u}'_{L,\alpha}\gamma^\mu d'_{L,\alpha} = \bar{u}_{L,\alpha}\gamma^\mu U^\dagger_{L,\alpha\beta} D_{L,\beta\gamma} d_{L,\gamma} = \bar{u}_{L,\alpha}\gamma^\mu d''_{L,\alpha} \tag{3.1}$$

where

$$d''_{L,\alpha} = U^\dagger_{L,\alpha\beta} D_{L,\beta\gamma} d_{L,\gamma} = V_{\alpha\gamma} d_{L,\gamma} \qquad (\alpha, \beta, \gamma = 1, \ldots, N)$$

There is no difficulty in passing U_L through γ^μ because the former matrix acts in flavor space, whereas the latter matrices act in the spin space (obviously we also use the *commutation* of complex numbers).

Thus the $Q = 1/3$ quark states participating in transitions of the charged weak current are linear combinations of mass eigenstates. The quark-mixing matrix V, being the product of two unitary matrices, is itself unitary.

An $N \times N$ unitary (complex) matrix is characterized by N^2 real-valued parameters. Of these, $N(N-1)/2$ are angles and $N(N+1)/2$ are phases. Not all the phases have physical significance, because $2N - 1$ of them can be removed by *quark rephasing* (Donoghue *et al.*, 1992). This leaves V with $(N-1)(N-2)/2$ such phases. Then, the unitary $N \times N$ (complex) matrix for N quark generations possesses $(N-1)^2$ observable real parameters. Obviously, in going from complex to quaternions, the calculation of the real-valued parameters of the CKM matrix drastically changes.

3.2. Complex-Barred Mixing Matrix

Working with the quaternionic field (the primes signify that the states which appear in the original gauge-invariant Lagrangian are generally *not* the mass eigenstates)

$$\psi'_{L,\alpha} = d'_{L,\alpha} + j u'_{L,\alpha} \qquad (\alpha = 1, \ldots, N)$$

we must expect to have $N \times N$ barred-quaternionic unitary matrices instead of complex unitary matrices.

Remembering that a barred-quaternion, in terms of real quantities, is expressed by

$$\mathfrak{Q} \equiv q + p \mid i \equiv \alpha_q + i\beta_q + j\gamma_q + k\delta_q + (\alpha_p + i\beta_p + j\gamma_p + k\delta_p) \mid i$$

with

$$\alpha_{q,p}, \ \beta_{q,p}, \ \gamma_{q,p}, \ \delta_{q,p} \ \in \ \mathfrak{R}$$

we have

$$\text{barred-quaternions} \supset \text{quaternions} \supset \text{complex}$$

and further

$$\text{barred-quaternions} \supset \text{barred-complex [elements like } \alpha + \beta \mid i \equiv \mathcal{C}]$$

We can now give the general formulas for counting the generators of unitary N-dimensional groups as a function of N:

$$U(N, \mathfrak{Q}): \quad 4N + 8 \frac{N(N-1)}{2} = 4N^2$$

$$U(N, q): \quad 3N + 4 \frac{N(N-1)}{2} = N(2N+1)$$

$$U(N, \mathcal{C}): \quad N + 2 \frac{N(N-1)}{2} = N^2$$

For a detailed discussion of quaternionic groups, the reader can consult De Leo and Rotelli (1996d).

Considering barred-quaternionic number as elements for our $N \times N$ unitary matrix, we quadruple the real-valued parameters; consequently we increase the real-valued parameter counting of the CKM matrix. This *puzzle* is soon overcome by noting that the mixing matrix has to commute with the gauge group $U_L(1, q) | U_Y(1, c)$. This restriction reduces to barred-complex the elements of the matrices which mix left-handed quarks. Finally, we have

$$\psi'_{L,\alpha} = D_{L,\alpha\beta}d_{L,\beta} + jU_{L,\alpha\beta}u_{L,\beta} \qquad (\alpha, \beta = 1, \ldots, N)$$

with

$$D_L, \ U_L \quad \text{unitary barred-complex matrices}$$

$$d_{L,R}, \ u_{L,R} \quad \text{complex mass eigenstates}$$

The transformation from the gauge basis states to the mass basis states turns out to have no effect on the structure of the electromagnetic and neutral weak currents. As example of this, consider the quark contribution to the weak currents:

$$\left\{ [\bar{d}_{L,\alpha}D^{\dagger}_{L,\alpha\beta} - \bar{u}_{L,\alpha}U^{\dagger}_{L,\alpha\beta}j] \right.$$

$$\left. \times \gamma_{\mu} \left[-\frac{g}{2} (i|W^{\mu}_1 + j|W^{\mu}_2 + k|W^{\mu}_3) \right] [D_{L,\beta\gamma}d_{L,\gamma} + jU_{L,\beta\gamma}u_{L,\gamma}] \right\}_c \quad (3.2)$$

After the complex projection we find no *flavor-changing neutral currents*; mixing between generations does manifest itself in the system of quark charged weak currents:

$$-\frac{g}{2} \bar{d}_{L,\alpha}D^{\dagger}_{L,\alpha\beta}\gamma_{\mu}(j|W^{\mu}_2 + k|W^{\mu}_3)jU_{L,\beta\gamma}u_{L,\gamma} + \text{h.c.}$$

and so

$$J^{\mu}_{ch} = \bar{d}_{L,\alpha}\gamma_{\mu}D^{\dagger}_{L,\alpha\beta}U_{L,\beta\gamma}u_{L,\gamma}(W^{\mu}_2 - iW^{\mu}_3) + \text{h.c.}$$

$$= \bar{d}_{L,\alpha}\gamma_{\mu}V_{\alpha\gamma}u_{L,\gamma}(W^{\mu}_2 - iW^{\mu}_3) + \text{h.c.} \quad (3.3)$$

where

$$V_{\alpha\gamma} = D^{\dagger}_{L,\alpha\beta}U_{L,\beta\gamma}$$

4. CONCLUSIONS

The primary interest of the author in recent years has been to demonstrate the possibility of using quaternions in the description of elementary particles.

The complex projection of scalar products and Lagrangians represents the fundamental ingredient in reformulating quaternionic quantum theories.

The noncommutative nature of quaternions makes the standard approach to the physical world complicated. A complex geometry seems necessary (if not sufficient) for reproducing standard quantum theories. In this work we have reviewed the quaternionic electroweak theory, based on the one-dimensional local gauge group $U_L(1, q) | U_Y(1, c)$ (minimal quaternionic unitary group for our Lagrangian) and overcome the apparent *puzzle* concerning the calculation of the real-valued parameters of the CKM matrix.

REFERENCES

Adler, S. L. (1986a). *Communications in Mathematical Physics*, **104**, 611.

Adler, S. L. (1986b). *Physical Review D*, **34**, 1871.

Adler, S. L. (1988). *Physical Review D*, **37**, 3654.

Adler, S. L. (1989). *Physics Letters*, **221B**, 39.

Adler, S. L. (1994a). *Physics Letters*, **332B**, 358.

Adler, S. L. (1994b). *Nuclear Physics B*, **415**, 195.

Adler, S. L. (1995). *Quaternionic Quantum Mechanics and Quantum Fields*, Oxford University Press, Oxford.

Adler, S. L. (1996a). Projective group representations in quaternionic Hilbert space, Preprint IASSNS-HEP 96/02 [hep-th/9601047].

Adler, S. L. (1996b). Quaternionic quantum mechanics and NONcommutative dynamics, Preprint IASSNS-HEP 96/72 [hep-th/9607008].

Adler, S. L., and Millard, A. C. (1996). Generalized quantum dynamics as pre-quantum mechanics, Preprint IASSNS-HEP 95/66 [hep-th/9508076].

Altmann, S. L. (1986). *Rotations, Quaternions, and Double Groups*, Clarendon Press, Oxford.

Cabbibo, N. (1963). *Physical Review Letters*, **10**, 531.

De Leo, S. (1996a). *Progress of Theoretical Physics*, **94**, 1109.

De Leo, S. (1996b). *International Journal of Modern Physics A*, **11** 3973.

De Leo, S. (1996c). *Journal of Mathematical Physics*, **37**, 2955.

De Leo, S. (1996d). Half-whole dimensions in quaternionic quantum mechanics, Preprint hep-th/9605055; *Progress of Theoretical Physics*, **95**, 1029.

De Leo, S., and Rotelli, P. (1994). *Progress of Theoretical Physics*, **92**, 917.

De Leo, S., and Rotelli, P. (1995a). *Nuovo Cimento B*, **110**, 33.

De Leo, S., and Rotelli, P. (1995b). *International Journal of Modern Physics A*, **10**, 4359.

De Leo, S., and Rotelli, P. (1996a). *Journal of Physics G*, **22**, 1137.

De Leo, S., and Rotelli, P. (1996b). Odd-dimensional translation between complex and quaternionic quantum mechanics, Preprint hep-th/9605021; *Progress of Theoretical Physics*, **96**, 247.

De Leo, S., and Rotelli, P. (1996c). *Modern Physics Letters A*, **11**, 357.

De Leo, S., and Rotelli, P. (1996d). *International Journal of Theoretical Physics*, **35**, 1821.

Dimitric, R., and Goldsmith, B. (1989). *Mathematical Intelligencer*, **11**, 29.

Donoghue, J. F., Golowich, E., and Holstein, B. R. (1992). *Dynamics of the Standard Model*, Cambridge University Press, Cambridge.

Finkelstein, D., *et al.* (1962). *Journal of Mathematical Physics*, **3**, 207.

Finkelstein, D., *et al.* (1963a). *Journal of Mathematical Physics*, **4**, 136.

Finkelstein, D., *et al.* (1963b). *Journal of Mathematical Physics,* **4,** 788.

Finkelstein, D., Jauch, J. M., and Speiser, D. (1979). Notes on quaternion quantum mechanics, in *Logico-Algebraic Approach to Quantum Mechanics II,* C. A. Hoker, ed., Reidel, Dordrecht, pp. 367–421.

Gilmore, R. (1974). *Lie Groups, Lie Algebras and Some of Their Applications,* Wiley, New York.

Glashow, S. L. (1961). *Nuclear Physics,* **22,** 579.

Hamilton, W. R. (1969). *Elements of Quaternions,* Chelsea, New York.

Horwitz, L. P. (1993). *Journal of Mathematical Physics,* **34,** 3405.

Horwitz, L. P., and Biedenharn, L. C. (1984). *Annals of Physics,* **157,** 432.

Kobayashi, M., and Maskawa, T. (1973). *Progress of Theoretical Physics,* **49,** 552.

Nash, C. C., and Joshi, G. C. (1987a). *Journal of Mathematical Physics,* **28,** 2883.

Nash, C. C., and Joshi, G. C. (1987b). *Journal of Mathematical Physics,* **28,** 2886.

Nash, C. C., and Joshi, G. C. (1988). *International Journal of Theoretical Physics,* **27,** 409.

Nash, C. C., and Joshi, G. C. (1993). *International Journal of Theoretical Physics,* **31,** 965.

Razon, A., and Horwitz, L. P. (1991). *Acta Applicande Mathematicae,* **24,** 141, 179.

Razon, A., and Horwitz, L. P. (1992). *Journal of Mathematical Physics,* **33,** 3098.

Rembieliński, J. (1978). *Journal of Physics A,* **11,** 2323.

Rembieliński, J. (1980a). *Journal of Physics A,* **13,** 15.

Rembieliński, J. (1980b). *Journal of Physics A,* **13,** 23.

Rotelli, P. (1989a). *Modern Physics Letters A,* **4,** 933.

Rotelli, P. (1989b). *Modern Physics Letters A,* **4,** 1763.

Salam, A. (1968). In *Proceedings of the 8th Nobel Symposium, Weak and Electromagnetic Interactions,* N. Svartholm, ed., Almqvist & Wiksell, Stockholm, and Wiley, New York, p. 367.

Weinberg, S. (1967). *Physical Review Letters,* **19,** 1264.